5 Week 5 - Spectral domain circuit analysis. Part 1

5.1 Background

Up to now we learned how to solve the response of a circuit to the switching on or switching off of a source of an otherwise constant source. As a segue to the case of switching on or off an source with an arbitrary function of time, we will consider the case of the steady-state response to a sinusoidal input. Steady-state means that we do not consider the transient response of the system after a switch, but only the response at the frequency of the driving sinusoid.

This problem can be solved in time. But it also can be solved much more simply in frequency space. Let's review the formalism of Fourier transforms.The transform from the time domain to the frequency domain is given by:

$$\begin{aligned} \tilde{V}(\omega) &\equiv \mathcal{F}(V(t)) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, V(t)\, e^{-i\omega t}. \end{aligned} \tag{5.5}$$

In essence we are projecting the time series $V(t)$ onto all frequencies ω and phases. The reverse transform is given by

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \tilde{V}(\omega)\, e^{i\omega t}. \tag{5.6}$$

Self-consistency leads to the definition of the delta function - a pulse of area one whose width goes to zero as its height goes to infinity. We have

$$\begin{aligned} \tilde{V}(\omega) &= \int_{-\infty}^{\infty} dt\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, \tilde{V}(\omega')\, e^{i\omega' t}\, e^{-i\omega t} \\ &= \int_{-\infty}^{\infty} d\omega'\, \tilde{V}(\omega')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i(\omega-\omega')t}. \end{aligned} \tag{5.7}$$

The delta function, denoted $\delta(\omega - \omega')$, is defined as

$$\delta(\omega - \omega') \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i(\omega-\omega')t}. \tag{5.8}$$

Physically, the complex exponent oscillates between plus one and minus one so that the integral goes to zero, except, when $\omega = \omega'$, for which the integral goes to infinity. To complete the analysis, we have

$$\begin{aligned} \tilde{V}(\omega) &= \int_{-\infty}^{\infty} d\omega'\, \tilde{V}(\omega')\, \delta(\omega - \omega') \\ &= \tilde{V}(\omega) \end{aligned} \tag{5.9}$$

The delta function can be defined as an integral over frequency

$$\delta(t-t') \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; e^{-i(t-t')\omega}. \tag{5.10}$$

The delta function must be used in the context of an integral; it does not stand by itself. Thus

$$\int_{-\infty}^{\infty} d\omega' \delta(\omega-\omega') = 1 \tag{5.11}$$

and

$$\int_{-\infty}^{\infty} d\omega' \delta(\omega-\omega')\tilde{f}(\omega') = \tilde{f}(\omega) \tag{5.12}$$

5.2 Utility

Let's see why this is useful. First, we need the representation of a sinusoid in frequency space. So let's take

$$V_{in}(t) = A\sin(\omega_D t). \tag{5.13}$$

Then

$$\begin{aligned}
\tilde{V}_{in}(\omega) &= A\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\; e^{-i\omega t}\sin(\omega_{\mathrm{D}}\mathrm{t}) \qquad (5.14)\\
&= A\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\; \frac{e^{i\omega_D t}-e^{-i\omega_D t}}{2i}\; e^{-i\omega t}\\
&= A\frac{1}{\sqrt{2\pi}}\frac{\pi}{i}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-i(\omega-\omega_D)t} \;-\; \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-i(\omega+\omega_D)t}\right]\\
&= A\;\frac{1}{i}\sqrt{\frac{\pi}{2}}\;\left[\delta(\omega-\omega_D)\;-\;\delta(\omega+\omega_D)\right].
\end{aligned}$$

Now we examine a second property of the Fourier transform - the transform of derivatives and integrals. Lets look at the Fourier transform of

$$\frac{dV_{in}(t)}{dt}. \tag{5.15}$$

Then

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\;\frac{dV_{in}(t)}{dt}\;e^{-i\omega t} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\frac{d\left(V_{in}(t)e^{-i\omega t}\right)}{dt} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dtV_{in}(t)(-i\omega)e^{-i\omega t}\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\left(V_{in}(t)e^{-i\omega t}\right) + i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dtV_{in}(t)e^{-i\omega t}\\
&= 0\;+\;i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dtV_{in}(t)e^{-i\omega t} \qquad (5.16)
\end{aligned}$$

where we must assume that the function $V_{in}(t)$ goes to zero at positive and negative infinity. So the Fourier transform of the time derivative is just the Fourier transform of the function multiplied by $i\omega$. This transforms differential equations into algebraic equations. In general,

$$\mathcal{F}\left(\frac{d^n V_{in}(t)}{dt^n}\right) = (i\omega)^n \tilde{V}_{in}(\omega). \tag{5.17}$$

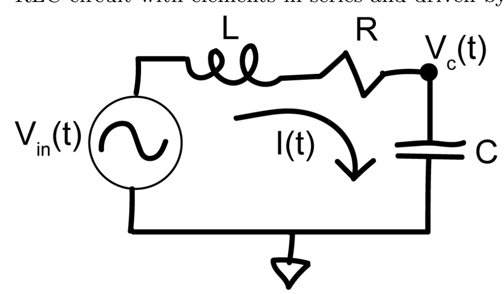
There is an analogous formula for integrals. We need to postpone the derivation, but

$$\mathcal{F}\left(\int_{-\infty}^{t} dt' V_{in}(t')\right) = \frac{1}{i\omega}\tilde{V}_{in}(\omega) + \pi\tilde{V}_{in}(0)\delta(\omega) \tag{5.18}$$

where the second term on the right is the just the average of the function. We can ignore this for now as long trains of sine and cosine have zero mean.

5.2.1 Impedance

Figure 1: RLC circuit with elements in series and driven by a sinusoid



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This formalism lets us rewrite the derivative and integral relations between current and voltage for capacitors and inductors in terms of algebra. The application of Kirchoff's voltage law yields

$$0 = -V_{in}(t) + L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}\int_{-\infty}^{t} dt'\ I(t'). \tag{5.19}$$

Fourier transforming yields

$$\tilde{V}_{in}(\omega) = +i\omega L\tilde{I}(\omega) + R\tilde{I}(\omega) + \frac{1}{i\omega C}\tilde{I}(\omega) \tag{5.20}$$

We see that we can define a frequency dependent "Resistance", denoted the Impedance, $\tilde{Z}(\omega)$, for capacitors and inductors. $\tilde{Z}(\omega)$ is in general a complex number. For capacitors,

$$I_C(t) = C\frac{dV_C(t)}{dt} \tag{5.21}$$

becomes

$$\tilde{I}_C(\omega) = i\omega C\tilde{V}_C(\omega) \tag{5.22}$$

so

$$\tilde{Z}_C(\omega) = \frac{1}{i\omega C}. \tag{5.23}$$

Similarly, for inductors

$$\tilde{Z}_L(\omega) = i\omega L. \tag{5.24}$$

Once you write components in terms of impedance, circuit analysis is algebraic and fast. Relations like voltage division can be calculated using impedances, and relations like parallel or serial resistance generalize for impedance.